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Abstract: Let S be a point set in the plane in general position, such that its elements are partitioned into k classes or *colors*. In this paper we study several variants on problems related to the Erdős-Szekeres Theorem about subsets of S in convex position, when additional chromatic constraints are considered.

Key-words: Computational geometry, Delaunay triangulation, randomization

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Variantes chromatiques du théorème de Erdős et Szekeres sur l'existence de configurations convexes dans un ensemble de points

Résumé : Pour un ensemble de points S en position générale dans le plan, partitionné en k *couleurs*, nous étudions l'existence de sous-ensemble de points en position convexe et vérifiant des contraintes sur les couleurs (monochromatique, hétérochromatique, polychromatique).

Mots-clés : Géométrie algorithmique, triangulation de Delaunay, randomisation

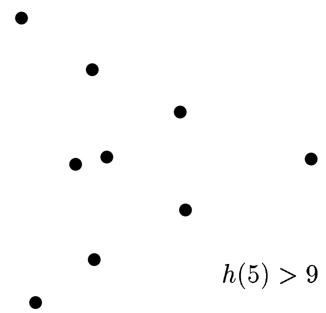

Figure 1: A set of 9 points with no 5-hole.

1 Introduction and preliminary results

The following result is commonly called *the Erdős-Szekeres Theorem*:

Theorem 1.1 [ES35] *For every natural number m there exists a number $n(m)$ such that any n-point set S in the plane in general position with $n \geq n(m)$ contains an m-subset of points in convex position.*

This problem has been attracting the attention of many researchers, both because its beauty and elementary statement, and because finding the exact value of $n(m)$ turns out to be a very challenging problem. The reader is referred to the survey paper [MS00] for a history of the problem, a description of many variants, and a wide list of references. The best currently known bounds are

$$2^{m-2} \leq n(m) \leq \binom{2m-5}{m-2} + 2,$$

where the lower bound was essentially proved by Erdős and Szekeres in their first papers and the upper bound is due to Tóth and Valtr [TV98].

Let A be a point set in the plane in general position. An m-point subset $B \subset A$ in convex position is called an *m-hole* in A if the convex hull $conv(B)$ is a polygon whose interior does not contain any point of A.

In 1978 Erdős [Erd78] raised the following problem: is there a number $h(m)$, for every natural m, such that any n-point set S in the plane in general position with $n \geq h(m)$ contains an m-hole?

Obviously $h(3) = 3$ and it is easy to see that $h(4) = 5$ and that $h(5) \geq 10$ (see Figure 1). Harborth [Har79] proved in 1978 that $h(5) = 10$, and in 1983 Horton [Hor83] showed that $h(m)$ does not exist for $m \geq 7$ by constructing arbitrarily large sets without a 7-hole. The existence of $h(6)$ is a problem that still remains open.

The seminal paper by Erdős and Szekeres already mentioned the generalization of their original problem to higher dimensions, but even in the plane many variants have been considered, we mention next some examples. Bistriczky and Fejes Tóth [BT89] proved a generalization replacing points with convex bodies. Bialostocki *et al* [ABV91], Caro [Car96] and Károlyi *et al* [KPTar] gave results on the conjecture from [ABV91] that for any given integers m and q any set large enough contains a n-set for which the number of interior points is divisible by q. Several authors have studied the number of subsets in convex position of a given size that a sufficiently large point set can have [BV98, Sol88, Nie95, Pur82, KM88, BF87, Val95, Dum00]. Let us finally mention the papers by Ambarcumjan [Amb66], Károlyi [Kár01], Hosono and Urabe [HU01] and Urabe [Ura96], where several issues on partitioning a point set into subsets in convex position are considered.

Let $S = S_1 \dot\cup \cdots \dot\cup S_k$ be a partition of a planar point set S in general position in the plane; we refer to S_i as the set of points of *color* i. A subset $T \subset S$ is called *monochromatic* if all its points have the same color, and *polychromatic* otherwise. The term *heterochromatic* is used for the special case in which every element in T has a different color.

In this paper we consider the following collection of problems. Given an integer m and a set S as above, possibly with additional requirements for $n = |S|$ to be large enough, can we find an m-hole of S falling into one of the three described chromatic classes? Or an m-subset in convex position?

The original motivation for us to study these problems came from a different area. A finite set Γ of curves in the plane is a *separator* for the sets $S_1, \ldots, S_k$ if every connected component in $\mathbb{R}^2 - \Gamma$ contains objects only from some S_i. We also say that each connected component is *monochromatic*. A thorough study of the subject is developed in [Seaon].

When we have two sets, say the *red* points and the *blue* points, another way to approach their separability is to look for triangulations in which as many edges as possible (or as many triangles as possible) are monochromatic, which somehow contributes to isolate the two populations.

As a consequence of the above motivation, in this paper we study the conditions for the existence of some configuration, and also consider how many *compatible* such configurations can we guarantee, where compatibility stands for having disjoint relative interiors. For example, if the configuration is a monochromatic edge, we also try to find how many monochromatic edges can we guarantee without producing any crossing; the compatibility allows the edges to be completed to a triangulation.

In the sequel we will study the numbers $n_M(m,k)$, the minimal number of points colored with k colors to ensure the existence of one monochromatic m-subset in convex position, and $MC(n,m,k)$, the minimal number of compatible monochromatic m-holes in a set of n points colored with k colors. $n_H(m,k)$, $n_P(m,k)$ $HC(n,m,k)$ and $PC(n,m,k)$ are similarly defined in the heterochromatic and polychromatic cases. A related (yet quite different) problem is considered in [NV94]. More in the spirit of our problems, several results are described in [NV98, NVS99, KPT97, KPTV98] but looking for configurations like cycles or paths when *edges* are colored.

2 Subsets in convex position

It us natural to start by considering subsets in convex position without the additional constraint of requiring empty interiors. We will see that the latter case, studied in the following sections, is much less simple.

Theorem 2.1

$$n_M(m,k) = k \cdot (n(m) - 1) + 1$$

Proof: The proof is straightforward since k copies with different colors of a set of size $n(m)-1$ without convex m-gons give a set without monochromatic m-gon. □

Theorem 2.2

$$\textit{if } k \geq n(m) \textit{ then } n_H(m,k) = n(m)$$

$$\textit{if } k < n(m) \textit{ then } n_H(m,k) = \infty$$

Proof: If $n \geq k \geq n(m)$ we extract an heterochromatic subset of size k and apply Erdős-Szekeres theorem to find an heterochromatic convex m-gon.

If $k < n(m)$ then we take a set of k points without convex m-gon and color the points with different colors. Then each point can be replaced by several points close together if they have the same color. □

The more interesting situation arises in the polychromatic case

Theorem 2.3

$$\textit{if } k \geq n(m) \textit{ then } n_P(m,k) = n(m)$$

$$\textit{if } k < n(m-1) \textit{ then } n_P(m,k) = \infty$$

Proof: If $k \geq n(m)$ the heterochromatic solution is a polychromatic solution.

If $k < n(m-1)$ then we take a set of k points without convex $(m-1)$-gon and color the points with different colors. Then a point on the convex hull can be replaced by a small piece of convex curve such that all the points see it from outside, then this curve can be sampled by many points on the same color such that only two of them can appear in a polychromatic convex subset. Finding a m-gon would contradict the hypothesis that the original set has no $(m-1)$-gon. □

There is a gap remaining for $n(m-1) \leq k < n(m)$. The examples of Figure 2 show that $n_P(5, n(4)) = n_P(5,5) = \infty$ and $n_P(6, n(5)) = n_P(6,9) = \infty$.

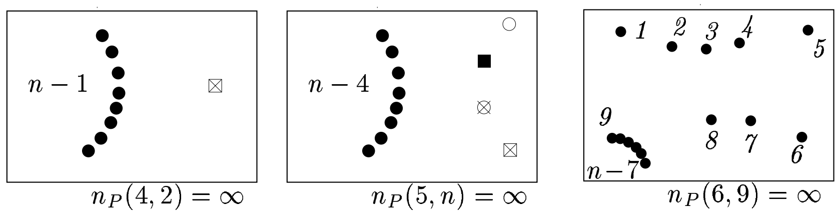

Figure 2: Polychromatic convex subsets.

3 Monochromatic holes

Theorem 3.1 *The minimum numbers of compatible monochromatic m-holes, $MC(n,m,k)$ are as follows:*

$m \setminus k$	*2*	*3*	
2	$\longleftarrow$	$2\lceil \frac{n}{k} \rceil - 3$	$\longrightarrow$
3	$\lceil \frac{n-8}{4} \rceil$	$0 \longrightarrow$	
4	*?*	$0 \longrightarrow$	
5	0	$0 \longrightarrow$	
6	0	$0 \longrightarrow$	
7	0	$0 \longrightarrow$	

The rest of this section is devoted to the proofs of these values.

3.1 Edges and triangles

The most frequent color has at least $\lceil \frac{n}{k} \rceil$ points and any triangulation of these points gives at least $2\lceil \frac{n}{k} \rceil - 3$ monochromatic edges. This is tight as shown by the following construction: take n points in convex position and color them *1,2,..., k,1,2,..., k..., t-1,t*; in this way we obtain several groups of consecutive points colored *1,2,..., k* and a final group colored *1,2,..., t*. As there is at most one monochromatic edge between two of these groups, the number of monochromatic edges cannot exceed the number of edges of a triangulation of $\lceil \frac{n}{k} \rceil$ points in convex position, which has $2\lceil \frac{n}{k} \rceil - 3$ edges.

$MC(n,3,2) \geq \lceil \frac{n-8}{4} \rceil$. Indeed this number of monochromatic triangles of the same color can be found in the following way. Assume without loss of generality that there are more red points than blue points, and let r' be the number of vertices of the convex hull of the red points, let r'' be the number of other red points, and let β be the number of blue points inside the red convex hull. On one hand, we consider any triangulation of the red points, such a triangulation has exactly $r' + 2r'' - 2$ triangles and trivially at most β of them contain some blue point, thus there are at least $r' + 2r'' - 2 - \beta$ empty compatible red triangles (notice that this number might be negative). On the other hand, we consider a triangulation of the β

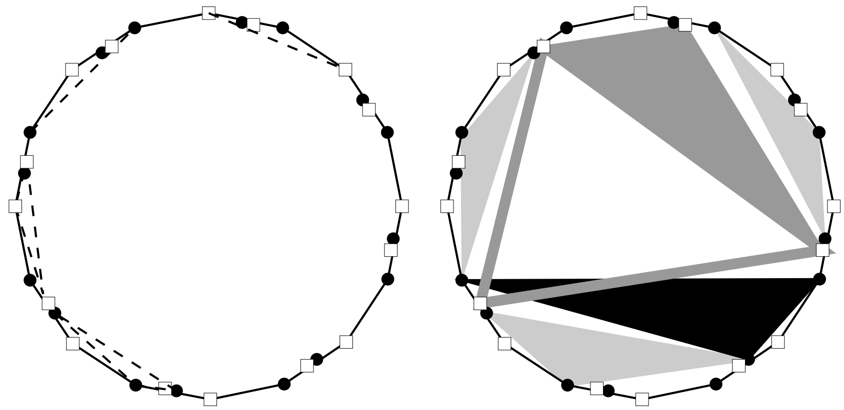

Figure 3: Tightness of $MC(n, 3, 2)$.

blue points inside the red convex hull, such a triangulation has at least $\beta - 2$ triangles (even more if the blue points are not in convex position) and at most r'' of them may contain a red point, thus there are at least $\beta - 2 - r''$ empty compatible blue triangles. The maximum of $r' + 2r'' - 2 - \beta$ and $\beta - 2 - r''$ is greater than $\frac{1}{2}(r' + 2r'' - 2 - \beta + \beta - 2 - r'') = \frac{r' + r'' - 4}{2} \geq \frac{n-8}{4}$.

This bound is tight, as shown by the example of Figure 3-left, this example is formed by a red convex polygon and a blue one rotated in such a way that only every second point on each polygon appears on the convex hull of the whole set. Let's take set maximal in size of monochromatic empty triangles, and let's consider a blue triangle T' and a red triangle T'' that see each other. It is always possible to remove the blue triangle T' and to create a new empty red triangle which replaces the blue one by linking the visible edge of the red triangle T'' to a red vertex neighboring the vertices of the removed blue triangle (see Figure 3-right). In this way we get a set of monochromatic empty triangles with maximal size, were all monochromatic empty triangles are red. The proof is easily finished now, because these triangles can be completed to a triangulation of the red points. Such a triangulation has $\lfloor \frac{n}{2} \rfloor - 2$ triangles and since $\lfloor \frac{n+1}{4} \rfloor$ edges of the red convex hull have a blue point very close to each of them, inside the hull, we have to remove this number of triangles since they are non empty and we get $\lfloor \frac{n}{2} \rfloor - 2 - \lfloor \frac{n+1}{4} \rfloor = \lceil \frac{n-8}{4} \rceil$ empty triangles.

3.2 Colored Horton sets

3.2.1 Preliminaries

The results of non existence of monochromatic m-holes is obtained by a appropriate coloring of the so called Horton set, which is an example of point set without 7-hole.

A Horton set [Hor83, MS00] is a set of n points sorted by x coordinates $p_1 <_x p_2 <_x p_3 <_x \ldots <_x p_n$ such that the odd points $p_1, p_3, \ldots$ and the even points $p_2, p_4, \ldots$ are Horton

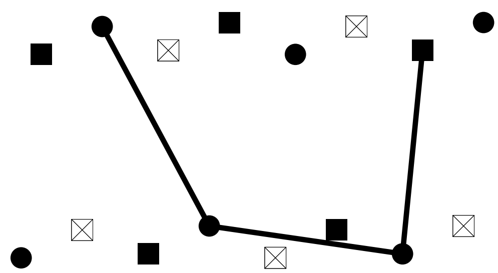

Figure 4: A Horton set of 16 points and a 4-cup.

sets and such that any line through two even points (the *upper set*) leaves all odd points below and any line through two odd points (the *lower set*) leaves all even points above. A Horton set of size n is recursively obtained by adding a large vertical separation after intertwining in the x direction an upper Horton set of size $\lfloor \frac{n}{2} \rfloor$ and a lower set of size $\lceil \frac{n}{2} \rceil$. Such a set is shown in Figure 4.

We define an r-cup (resp. r-cap) as a subset of r points in convex position $p_{i_1} <_x p_{i_2} <_x \ldots <_x p_{i_r}$ such that the upper (resp. lower) convex hull is the segment $p_{i_1}p_{i_r}$. If no other points with abscissa between p_{i_1} and p_{i_r} are inside the convex hull or above it, we say that the cup is empty. An empty cap is defined similarly.

A Horton set has the property that any 4-cap (and any 4-cup) is always non-empty [Hor83, MS00] (see Figure 4).

Then the non-existence of an 7-hole in the Horton set is easy to prove. An heptagon having only odd or only even points is non empty by an induction hypothesis. Any heptagon with some odd points and some even points must have at least four of the same kind, say odd without loss of generality, and these four points would define a 4-cup of the odd Horton subset (considered alone), therefore it has an odd point above which makes the cup non-empty; this point is necessarily inside the heptagon, otherwise the property that a line through odd points leaves all even points above would be violated.

3.2.2 Triangles in trichromatic set

Let us first observe that in a Horton set, the indices of the vertices of any empty 2-cup (same for an empty 2-cap) differ by a power of two. Indeed, the two vertices of the cup cannot be both odd, because an even point would always prevent the emptyness. If one point is odd and one is even, their indices can only differ by one, and if both vertices are even, they differ by a power of two by induction on the upper Horton set, when considered alone, and the difference of the indices gets multiplied by 2 once the odd points are recovered.

We are now ready to prove that $MC(n,3,3) = 0$. Consider the Horton set and color the points with the three colors R, G and B so that the points are colored $RGBRGBRGB\ldots$ in the x order. This coloration splits well recursively, indeed the upper Horton set is colored $GRBGRB\ldots$ and the lower Horton set is colored $RBGRBG\ldots$ (see Figure 4).

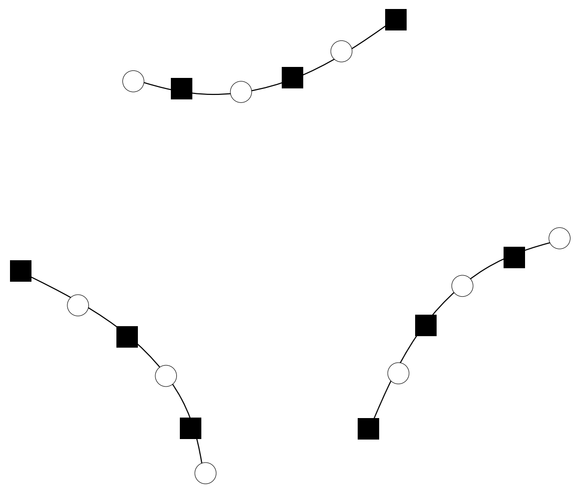

Figure 5: A set of 18 points with no monochromatic 4-hole.

This set does not contain an empty monochromatic 2-cup (or 2-cap) because by the above remark the vertices would have indices different by a power of two, but this disallows that they have the same color, as a power of two cannot be a multiple of 3. Therefore we cannot get monochromatic triangles where two vertices belong to the upper set and one to the lower one, or reversely.

Triangles made by three points all from the upper (or lower) part, are not monochromatic by inductive hypothesis, and this completes the proof.

3.2.3 Pentagon in bichromatic set

The fact that $MC(n,5,2) = 0$ is a variation of the previous result. We use the same 3-coloring, but we transform it into a coloring with only two colors by identifying colors G and B as single color denoted GB. An empty monochromatic pentagon for the two colors R and GB can be either of color R or of color GB, in the second case at least three vertices of the pentagon have the same color with respect to the original 3-coloring; in all cases we can extract an empty monochromatic triangle from the pentagon which contradicts our previous result that $MC(n,3,3) = 0$, and thus an empty monochromatic pentagon does not exist in this 2-coloring.

3.3 Open problem: $k = 2$, $m = 4$

The problem of finding $MC(n,4,2)$ remains open, however we can remark that a bichromatic set with no monochromatic 4-hole cannot contain any 7-hole, otherwise such heptagon would have at least 4 points of the same color and they would form a monochromatic 4-hole. As a consequence, any example showing $MC(n,4,2)=0$ must be an Horton set or an equivalent construction.

A construction with 18 points which contain no monochromatic 4-hole is shown on Figure 5 proving that $MC(n,4,2)=0$ for $n \leq 18$.

Conjecture 3.1 *For n large enough, $MC(n,4,2)>0$. In other words, every large bichromatic point set contains some monochromatic 4-hole.*

4 Heterochromatic holes

Theorem 4.1 *The minimum numbers of compatible heterochromatic m-holes, $MH(n,m,k)$ are the following:*

<table>
<tr><td>$m \setminus k$</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td></td></tr>
<tr><td>2</td><td colspan="7">$\longleftarrow$ $n+k-3$ $\longrightarrow$</td></tr>
<tr><td>3</td><td></td><td colspan="6">$\longleftarrow$ $k-2$ $\longrightarrow$</td></tr>
<tr><td>4</td><td></td><td></td><td colspan="5">0 $(\forall n \geq 2k-3)$</td></tr>
<tr><td>5</td><td></td><td></td><td></td><td colspan="4">0 $(\forall n \geq 2k-4)$</td></tr>
<tr><td>6</td><td></td><td></td><td></td><td></td><td colspan="3">0 $(\forall n \geq 2k-5)$</td></tr>
<tr><td>7</td><td></td><td></td><td></td><td></td><td></td><td colspan="2">0 $\longrightarrow$</td></tr>
</table>

The remain of the section contains the proofs of the above values.

4.1 Edges and Triangles

Consider a set of n points in convex position such that the points with the same color appear consecutively on the convex hull. Any triangulation of such a set has $2n-3$ edges but $n-k$ edges of the convex hull are monochromatic, thus the number of heterochromatic edges cannot exceed $n-k-3$. In that triangulation, the number of triangles is $n-2$ and among them the $n-k$ triangles incident to the $n-k$ monochromatic edges of the convex hull are not heterochromatic, thus the number of heterochromatic triangles cannot exceed $k-2$.

To achieve such bound for triangles, we can construct any triangulation of k points with different colors, this gives at least $k-2$ different triangles which are all obviously heterochromatic. Now the remaining points are inserted in turn. If a point falls inside a triangle, this triangle is split into 3 triangles by linking the point to the vertices, if it falls outside the convex hull, the point is linked to all visible points of the convex hull. In this algorithm, the number of starting heterochromatic triangles can never decrease, because when an heterochromatic triangle disappears because a point is inserted inside it, at least one of the three newly created triangles is heterochromatic.

To achieve the bound for edges, we take a point p of color *1* and link all other points to it creating $n-n_1$ heterochromatic edges (where n_1 is the number of points of color *1*); then enumerating these edges in polar order around p, there are at least $k-2$ changes of color, for each change, we add the edges linking these pair of consecutive points creating $k-2$ new heterochromatic edges; finally, the remaining $n_1 - 1$ points always see a point of another color and can be linked to it. Thus we get a total of $n - n_1 + k - 2 + n_1 - 1 = n + k - 3$ edges matching the above lower bound.

4.2 m-holes, $m \geq 4$

First notice that for $m \geq 7$ we cannot expect heterochromatic 7-holes since there are sets with no 7-holes at all.

If $n \geq 2k - m + 1$, the construction of Figure 6 describes a point set without heterochromatic m-holes. A point of each color is placed on a convex curve strictly decreasing (as a function), then, close enough to points of all colors from *m-1* to *k-1*, a point of color *1* (the *obstacle point*) is placed at the same ordinate and nearly the same abscissa (increased by a small ϵ), in such a way that the subset of the points with color *1, 2, m-1* and k form still a convex curve and that any chord enclosing a point of color between *m-1* and *k-1* encloses the corresponding point of color *1*. In this construction we have $n = 2k - m + 1$, any greater value of n can be reached by adding points of color l $(m - 1 \leq l \leq k - 1)$ between the point with color l and its sibling obstacle point with color *1*.

For small values of $n-k$ and large k there is always some heterochromatic k-hole provided that $h(m) < \infty$. They can be constructed by taking a point p_k on the convex hull and sorting $k-1$ points of different colors by polar order around it, which we denote by $p_0, p_1 \ldots p_{k-1}$; then we can get $\lfloor (k - h(m) + 3)/(h(m) - 2) \rfloor$ subsets

$$p_k, p_{(h(m)-1)i}, p_{(h(m)-1)i+1}, \ldots, p_{(h(m)-1)i+(h(m)-1)},$$

for $0 \leq i < \lfloor \frac{k-h(m)+3}{h(m)-2} \rfloor$, whose convex hulls are disjoint. By the usual result on non chromatic holes we can extract an m-hole from each of these sets, and all them are clearly heterochromatic. By adding the $n-k$ remaining points at most $n-k$ of these m-gons can become non empty and we get that $HC(n, m, k) \geq \lfloor \frac{k-h(m)+3}{h(m)-2} \rfloor + k - n$ which is positive for $h(m) \leq k \leq n \leq \lfloor \frac{h(m)(k-1)+3}{h(m)-2} \rfloor$.

For $\lfloor \frac{h(m)(k-1)+3}{h(m)-2} \rfloor \leq n < 2k - m + 1$ there is a gap in our results in which we do not know whether $HC(n, m, k)$ is non-zero. In Figure 6 it is shown that $HC(4, 4, 4)$ and $HC(6, 4, 5)$ are zero.

5 Polychromatic holes

Theorem 5.1 *The minimum numbers of compatible polychromatic m-holes,* $MP(n, m, k)$*, are the following:*

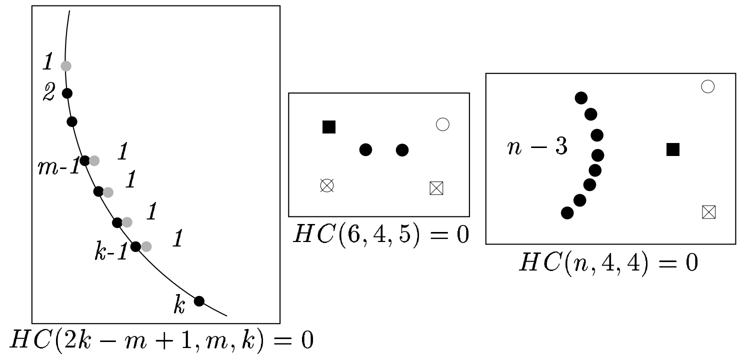

Figure 6: Counter examples to the existence of heterochromatic 4-hole.

$m \setminus k$	*2*	*3*	*4*	5	*6*	*7*	*8*	*9*	*10*	*11*	
2	$\longleftarrow$ $n+k-3$ $\longrightarrow$										
3	$\longleftarrow$ $n-2$ $\longrightarrow$										
4	0	1	2	$\geq \lfloor \frac{k-2}{3} \rfloor$							
5	0	0	0	0	*?*	*?*	*?*	*?*	*?*	*?*	
6								$\longleftarrow$	0	*?*	
7	0	$\longrightarrow$									

The proofs for the above results follow. Notice that in the case $m = 2$ polychromatic segments and heterochromatic segments are exactly the same concepts, and the result comes directly from the preceding section.

5.1 Polychromatic triangles

We follow the construction shown in the Figure 7. We choose a point p of color *1* and link it to all points with different colors, this partition the plane in sectors with apex p (Figure 7-left). In every sector qpr, link all the points with color *1* to q and link them in their radial order around q (Figure 7-center); in this way we obtain $n_1 - 1$ polychromatic triangles (where n_1 is the total number of points with color *1*). Finally, in sector qpr link r and p to a point which is visible to both r and p (Figure 7-right), creating $n - n_1 - 1$ additional polychromatic triangles over all sectors (or $n - n_1$ if p is not an extreme point). Thus we get a total of at least $n-2$ polychromatic triangles. This bound is tight, as shown by the construction giving $PC(n,4,2) = 0$ on Figure 8.

5.2 m-holes for $m \geq 4$

First of all, let us notice that for $m \geq 7$ we can not expect polychromatic 7-holes since there are sets with no 7-holes at all.

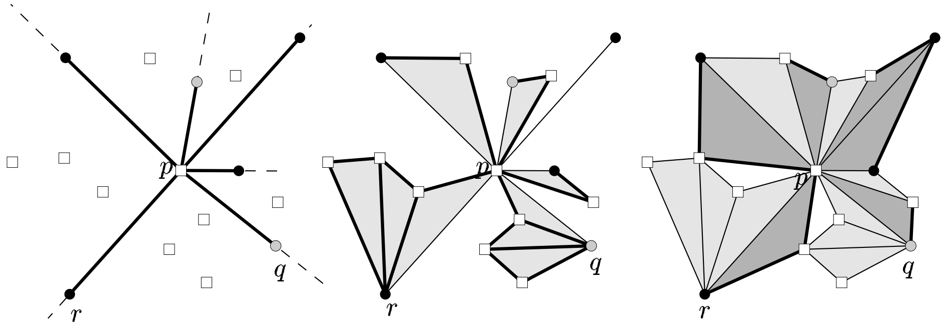

Figure 7: Triangulating with polychromatic triangles.

We first prove that $PC(n,4,3) = 1$ for $n \geq 5$. Let us take a point p of the less popular color, and sort the points radially around p; this gives necessarily a sequence of four consecutive points having less than 2 points of the same color as p and having at least two different colors. Together with p, we thus have a set of 5 points having no other points in their convex hull; therefore, since $h(4) = 5$, there is a 4-hole there, and since there are at most three points of the same color in the 5 points, the 4-hole is polychromatic. This is tight as shown on Figure 8.

To prove that $PC(n,4,k) \geq \lfloor \frac{k-2}{3} \rfloor$, we take an heterochromatic subset of size k and sort this points by polar order around a point p on the convex hull. Taking points 4 by 4 with an overlap of 1 gives $\lfloor \frac{k-2}{3} \rfloor$ heterochromatic subsets of size 5 with disjoint convex hulls, then we can add the other points and we have $\lfloor \frac{k-2}{3} \rfloor$ subsets of at least 5 points containing each not less than 5 colors. Using the fact that $PC(n,4,3) = 1$ in each sector yields to the result.

The examples of Figure 8 show some examples proving that $PC(n,4,2) = 0$, $PC(n,5,k) = 0$ for $k \leq 5$ and that $PC(n,6,k) = 0$ for $k \leq 10$.

6 Conclusion

Several results on a generalization of the Erdős-Szekeres Theorem to colored sets of points have been presented in this paper.

The chromatic version with k colors differs significantly from the non-chromatic version since for fixed m and k and n large at will, it is possible to construct point sets with no heterochromatic or polychromatic subset of size m in convex position.

Results are more interesting for the problem of the existence of m-holes for $3 \leq m \leq 6$. We have succeeded in proving some results on the existence or non-existence of m-holes, but some intriguing problems remain open. Among them, our conjecture on the existence of a monochromatic 4-hole in any large enough bichromatic point set is maybe the most challenging one.

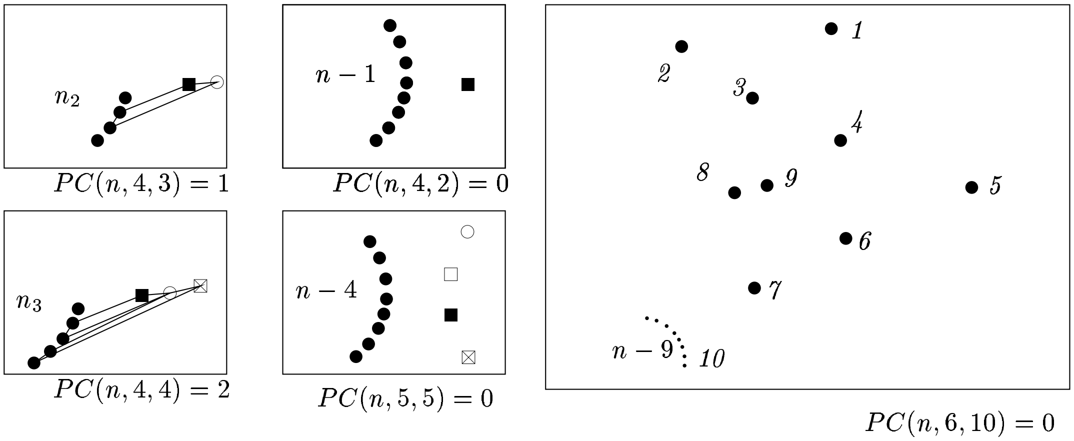

Figure 8: Sets of n points with no polychromatic m-hole, and tightness of $PC(n,4,3)=1$.

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